III. FOURIER TRANSFORM ON $L^2(\mathbb{R})$

In this chapter we will discuss the Fourier transform of Lebesgue square integrable functions defined on $\mathbb{R}$. To fix the notation, we denote

$$L^2(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \mid \int_{-\infty}^{\infty} |f(t)|^2 dt < \infty\}.$$

Unlike $L^1(\mathbb{R})$ we have discussed in the last chapter, $L^2(\mathbb{R})$ is a Hilbert space with an inner product defined as follows: For any $f, g \in L^2(\mathbb{R})$,

$$\langle f, g\rangle_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} f(t)\overline{g(t)} dt.$$

Note that $\langle f, g\rangle_{L^2(\mathbb{R})}$ is well-defined for any $f, g \in L^2(\mathbb{R})$, since in the case of $L^2(\mathbb{R})$, **Hölder's inequality** guarantees that for any $f, g \in L^2(\mathbb{R})$,

$$\int_{-\infty}^{\infty} \mid f(t)g(t) \mid dt \le (\int_{-\infty}^{\infty} \mid f(t) \mid^2 dt)^{\frac{1}{2}} \cdot (\int_{-\infty}^{\infty} \mid g(t) \mid^2 dt)^{\frac{1}{2}}.$$

The norm $|| \cdot ||_{L^2(\mathbb{R})}$ induced by $\langle f, g\rangle_{L^2(\mathbb{R})}$ has the following form:

$$||f||_{L^2(\mathbb{R})} = (\int_{-\infty}^{\infty} |f(t)|^2 dt)^{\frac{1}{2}}.$$

For convenience, we usually write $||\cdot||_2$ in stead of $||\cdot||_{L^2(\mathbb{R})}$. The concepts of **Cauchy sequence** in $L^2(\mathbb{R})$ and **convergence of sequence of functions** in $L^2(\mathbb{R})$ can be defined in exactly the same way as we have done for general Hilbert space. Also, $L^2(\mathbb{R})$ being a Hilbert space, every Cauchy sequence in $L^2(\mathbb{R})$ converges to some function in $L^2(\mathbb{R})$. Since $L^2(\mathbb{R})$ is not contained in $L^1(\mathbb{R})$, there are functions in $L^2(\mathbb{R})$ that do not belong to $L^1(\mathbb{R})$. For an arbitrary function $f \in L^2(\mathbb{R})$,

$$\int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$$

may not even be well defined. Our goal in this chapter is to figure out a way of defining the Fourier transform for all functions in $L^2(\mathbb{R})$, and find some of its properties. We begin by looking at the functions in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and finding out some useful properties of their Fourier transform. We say that a subset S of a Hilbert space $\mathcal{H}$ is **dense** in $\mathcal{H}$ if for any vector $x \in \mathcal{H}$, there is a sequence $\{x_n\}_{n=1}^{\infty} \subset S$ such that $\{x_n\}_{n=1}^{\infty}$ converges to x under the norm of $\mathcal{H}$. The reason behind our approach is the following fact:

Theorem 1. *$L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$.*

Proof. For any $f \in L^2(\mathbb{R})$, we only need to find a sequence of functions $\{f_n\}_{n=1}^{\infty} \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, such that $||f - f_n||_2 \longrightarrow 0$ as $n \to \infty$. If for every $n \in \mathbb{N}$, we let

$$f_n(x) = \begin{cases} f(x) & |x| \leq n \\ 0 & |x| > n \end{cases}$$

Then for each $n \in \mathbb{N}$, since

$$\int_{-\infty}^{\infty} |f_n(x)| dx = \int_{-\infty}^{\infty} |f(x)| \chi_{[-n,n]}(x) dx$$

$$\leq (\int_{-\infty}^{\infty} |f(x)|^2 dx)^{\frac{1}{2}} \cdot (\int_{-\infty}^{\infty} |\chi_{[-n,n]}(x)|^2 dx)^{\frac{1}{2}} = \sqrt{2n} ||f||_2 \leq \infty,$$

$$\int_{-\infty}^{\infty} |f_n(x)|^2 dx \leq \int_{-\infty}^{\infty} |f(x)|^2 dx \leq \infty,$$

so $\{f_n\}_{n=1}^{\infty} \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Also since for all $x \in \mathbb{R}$,

$$\lim_{n \to \infty} |f(x) - f_n(x)|^2 = 0,$$

$$|f(x) - f_n(x)|^2 \leq 2(|f(x)|^2 + |f_n(x)|^2) \leq 4|f(x)|^2$$

and $4|f|^2 \in L^1(\mathbb{R})$, so by Lebesque Dominant Convergence Theorem,

$$\lim_{n \to \infty} ||f - f_n||_2^2 = \lim_{n \to \infty} \int_{-\infty}^{\infty} |f(x) - f_n(x)|^2 dx = \int_{-\infty}^{\infty} (\lim_{n \to \infty} |f(x) - f_n(x)|^2) dx = 0. \quad \square$$

Now we proceed to investigate the Fourier transform of functions in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. But first we need some lemmas.

Lemma 1. *For $f \in L^2(\mathbb{R})$, let $F(t) = \int_{-\infty}^{\infty} f(u)\overline{f(t+u)} du$. Then*

(a) For any $t \in \mathbb{R}$, $|F(t)| \leq ||f||_2^2$.

(b)$F(t)$ is uniformly continuous on $\mathbb{R}$.

Proof. (a) We compute, for any $t \in \mathbb{R}$,

$$|F(t)| = |\int_{-\infty}^{\infty} f(u)\overline{f(t+u)} du| \leq \int_{-\infty}^{\infty} |f(u)| \cdot |f(t+u)| du$$

$$\leq (\int_{-\infty}^{\infty} | f(u) |^2 du)^{\frac{1}{2}} \cdot (\int_{-\infty}^{\infty} | f(t+u) |^2 du)^{\frac{1}{2}} = ||f||_2 \cdot ||f||_2.$$

(b) Similarly, for any $t, \eta \in \mathbb{R}$,

$$|F(t+\eta) - F(t)| \leq \int_{-\infty}^{\infty} |f(u)| \cdot |f(t+u+\eta) - f(t+u)| du$$

$$\leq \left(\int_{-\infty}^{\infty} \mid f(u) \mid^2 du\right)^{\frac{1}{2}} \cdot \left(\int_{-\infty}^{\infty} \mid f(t+u+\eta) - f(t+u) \mid^2 du\right)^{\frac{1}{2}}$$

$$= ||f||_2 \cdot \left(\int_{-\infty}^{\infty} \mid f(u+\eta) - f(u) \mid^2 du\right)^{\frac{1}{2}}.$$

Now according to some theorem in Lebesque integration,

$$\lim_{\eta \to 0} \int_{-\infty}^{\infty} \mid f(u+\eta) - f(u) \mid^2 du = 0,$$

where the limit is independent of specific $t \in \mathbb{R}$, so the conclusion follows. □

We change some conditions in Proposition 1 of last chapter to adapt to our needs in this chapter. The proof follows the same line and is slightly simpler in certain places. We leave the proof for the reader.

Proposition 1. *Let f be a function bounded on $\mathbb{R}$. $f(x)$ is continuous at $x = t$. Then*

$$\lim_{\alpha \to 0^+} (f * G_\alpha)(t) = f(t).$$

We also need an important lemma in Lebesque integration. The following is one of its simplified version.

Fatou's Lemma. *Suppose that a sequence of Lebesque integrable functions $\{f_n\}_{n=1}^{\infty}$ satisfies the following conditions:*

(a) $f_n(x) \geq 0$ for every $n \in \mathbb{N}$ and every $x \in \mathbb{R}$.

(b) $\lim_{n\to\infty} f_n(x)$ exists for every $x \in \mathbb{R}$.

(c) $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx$ exists. Then

$$\int_{-\infty}^{\infty} (\lim_{n\to\infty} f_n(x))dx \leq \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx.$$

Theorem 2. *Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then $\hat{f} \in L^2(\mathbb{R})$. Furthermore $||\hat{f}||_2^2 = 2\pi||f||_2^2$.*

Proof. Since $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, by Theorem 2 in last Chapter, $|\hat{f}(\xi)|^2 \leq ||f||_1^2$ for all $\xi \in \mathbb{R}$. Also, for Gaussian function G_α, $\widehat{G_\alpha}(\xi) = e^{-\alpha\xi^2}$ by Lemma 2 of last chapter, so $\widehat{G_\alpha} \in L^1(\mathbb{R})$. Hence as the product of a bounded function with a $L^1(\mathbb{R})$ function, $\widehat{G_\alpha}|\hat{f}|^2 \in L^1(\mathbb{R})$. If we denote $I = \int_{-\infty}^{\infty} \widehat{G_\alpha}(x)|\hat{f}(x)|^2dx$, then

$$I = \int_{-\infty}^{\infty} \widehat{G_\alpha}(x) \cdot \hat{f}(x) \cdot \overline{\hat{f}(x)}dx$$

$$= \int_{-\infty}^{\infty} \widehat{G_\alpha}(x) \cdot (\int_{-\infty}^{\infty} f(u)e^{-iux}du) \cdot (\int_{-\infty}^{\infty} \overline{f(v)e^{-ivx}dv})dx$$

Now we start to change the order of integration, this process is justified by a triple integral version of **Fubini's Theorem** and the fact that

$$\int_{-\infty}^{\infty} \widehat{G_\alpha}(x) \cdot (\int_{-\infty}^{\infty} |f(u)e^{-iux}|du) \cdot (\int_{-\infty}^{\infty} |\overline{f(v)e^{-ivx}}|dv)dx \leq ||f||_1^2 \int_{-\infty}^{\infty} \widehat{G_\alpha}(x)dx < \infty.$$

We change the order in such a way that we integrate with respect to x first, then v, and lastly u. Then by applying Theorem 4 of last chapter to G_α (note that G_α is a continuous function, so Theorem 4 of last chapter applies), and making a change of variable by letting $t = v - u$, and changing the order of integration again to integrate with respect to u first and t next (why is the change of order justified?), we can write

$$I = 2\pi \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} f(u)\overline{f(t+u)}du)G_\alpha(t)dt.$$

If we use the notation in Lemma 1 to denote $F(t) = \int_{-\infty}^{\infty} f(u)\overline{f(t+u)}du$, then according to Lemma 1, F is bounded and continuous on $\mathbb{R}$. And

$$I = 2\pi \int_{-\infty}^{\infty} F(t)G_\alpha(t)dt = 2\pi \int_{-\infty}^{\infty} F(t)G_\alpha(0-t)dt = 2\pi(F * G_\alpha)(0).$$

Now by applying Proposition 1, we get

$$\lim_{\alpha \to 0^+} \int_{-\infty}^{\infty} \widehat{G_\alpha}(x)|\hat{f}(x)|^2 dx = \lim_{\alpha \to 0^+} 2\pi(F * G_\alpha)(0) = 2\pi F(0) = 2\pi ||f||_2^2 < \infty.$$

Consequently, for any sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers with $\lim_{n\to\infty} \alpha_n = 0$, we have

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \widehat{G_{\alpha_n}}(x)|\hat{f}(x)|^2 dx = 2\pi ||f||_2^2,$$

On the other hand, for all $x \in \mathbb{R}$, $\lim_{n\to\infty} \widehat{G_{\alpha_n}}(x) = \lim_{n\to\infty} e^{-\alpha_n x^2} = 1$. Now Fatou's Lemma would imply that $\hat{f} \in L^2(\mathbb{R})$. Indeed, we have that

$$\int_{-\infty}^{\infty} |\hat{f}(x)|^2 dx = \int_{-\infty}^{\infty} \lim_{n \to \infty} \widehat{G_{\alpha_n}}(x)|\hat{f}(x)|^2 dx$$

$$\leq \lim_{n \to \infty} \int_{-\infty}^{\infty} \widehat{G_{\alpha_n}}(x)|\hat{f}(x)|^2 dx = 2\pi ||f||_2^2 < \infty.$$

Since $f \in L^2(\mathbb{R})$, so $|\hat{f}(x)|^2 \in L^1(\mathbb{R})$. Recalling the Gaussian functions discussed in Chapter 2, we see that $\widehat{G_{\alpha_n}}(x)|\hat{f}(x)|^2 = e^{-\alpha_n x^2}|\hat{f}(x)|^2 \leq |\hat{f}(x)|^2$. Hence,

$$\int_{-\infty}^{\infty} |\hat{f}(x)|^2 dx = \int_{-\infty}^{\infty} \lim_{n \to \infty} \widehat{G_{\alpha_n}}(x)|\hat{f}(x)|^2 dx = \lim_{n \to \infty} \int_{-\infty}^{\infty} \widehat{G_{\alpha_n}}(x)|\hat{f}(x)|^2 dx = 2\pi ||f||_2^2.$$

by Lebesque Dominant Convergence Theorem. $\square$

Remark From the proof of Theorem 1, given any $f \in L^2(\mathbb{R})$, we can find a sequence of functions $\{f_n\}_{n=1}^{\infty} \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ as defined in the same proof, such that $\lim_{n\to\infty} ||f - f_n||_2^2 = 0$. This implies that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(\mathbb{R})$. Now for any $m, n \in \mathbb{N}$, certainly $f_m - f_n \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, also $\hat{f}_m - \hat{f}_n = \widehat{f_m - f_n}$, so according to Theorem 2, $||\hat{f}_m - \hat{f}_n||_2^2 = ||\widehat{f_m - f_n}||_2^2 = 2\pi ||f_m - f_n||_2^2$. This implies that $\{\hat{f}_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in $L^2(\mathbb{R})$, hence there is a function $g \in L^2(\mathbb{R})$, such that $\{\hat{f}_n\}_{n=1}^{\infty}$ converges to g under the norm of $L^2(\mathbb{R})$.

Moreover, suppose $\{h_n\}_{n=1}^{\infty}$ is another Cauchy sequence in $L^2(\mathbb{R})$ that converges under the norm of $L^2(\mathbb{R})$ to the same given $f \in L^2(\mathbb{R})$, then $\{\hat{h}_n\}_{n=1}^{\infty}$ converges under the norm of $L^2(\mathbb{R})$ to the same g. Indeed, assume that $\{\hat{h}_n\}_{n=1}^{\infty}$ converges under the norm of $L^2(\mathbb{R})$ to some function $g' \in L^2(\mathbb{R})$, then

$$||g - g'||_2 \leq ||g - \hat{f}_n + \hat{f}_n - \hat{h}_n + \hat{h}_n - g'||_2$$

$$\leq ||g - \hat{f}_n||_2 + ||\hat{f}_n - \hat{h}_n||_2 + ||\hat{h}_n - g'||_2$$

$$= ||g - \hat{f}_n||_2 + 2\pi ||f_n - h_n||_2 + ||\hat{h}_n - g'||_2.$$

Now using the fact that $\{\hat{f}_n\}_{n=1}^{\infty}$ and $\{\hat{h}_n\}_{n=1}^{\infty}$ converges under the norm of $L^2(\mathbb{R})$ to the same $f \in L^2(\mathbb{R})$, for any $\varepsilon > 0$, we can show that $||g - g'||_2 < \varepsilon$, hence $g = g'$. Details are left for the reader.

Definition 1. *For any $f \in L^2(\mathbb{R})$, we define the Fourier transform of f as the limit of the sequence $\{\hat{f}_n\}_{n=1}^{\infty}$ under the norm of $L^2(\mathbb{R})$, where $\{f_n\}_{n=1}^{\infty} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is any sequence converges to f under the norm of $L^2(\mathbb{R})$. We also use the notation $\mathcal{F}(f)$ or $\hat{f}$ to denote such function g.*

Now let us briefly discuss some properties of Fourier transform of functions in $L^2(\mathbb{R})$.

Theorem 3. Parseval's Identity *For any $f, g \in L^2(\mathbb{R})$, we have $\langle f, g\rangle = \frac{1}{2\pi}\langle \hat{f}, \hat{g}\rangle$. In particular, $||f||_2 = \sqrt{\frac{1}{2\pi}}||\hat{f}||_2$.*

Proof. We prove the second conclusion first. To this end, we take any $\{f_n\}_{n=1}^{\infty} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ convergent to f under the norm of $L^2(\mathbb{R})$, then by Theorem 2, for each $n \in \mathbb{N}$, $||f_n||_2 = \sqrt{\frac{1}{2\pi}}||\hat{f}_n||_2$. Hence by triangle inequality of absolute value of real numbers and triangle inequality of $L^2(\mathbb{R})$ norm, we have

$$\left|||f||_2 - \sqrt{\frac{1}{2\pi}}||\hat{f}||_2\right| \leq \left|||f||_2 - ||f_n||_2\right| + \left|\sqrt{\frac{1}{2\pi}}||\hat{f}_n||_2 - \sqrt{\frac{1}{2\pi}}||\hat{f}||_2\right|$$

$$\leq ||f - f_n||_2 + \sqrt{\frac{1}{2\pi}}||\hat{f}_n - \hat{f}||_2.$$

Now for any $\varepsilon > 0$, we can show that $\left| ||f||_2 - \sqrt{\frac{1}{2\pi}}||\hat{f}||_2 \right| < \varepsilon$, hence $||f||_2 = \sqrt{\frac{1}{2\pi}}||\hat{f}||_2$. Details are left for the reader. Once it is done, the first conclusion can then be deduced with the help Lemma 2 below. $\square$

Lemma 2. *For any $f, g \in L^2(\mathbb{R})$,*

$$\langle f, g\rangle = \frac{||f+g||_2^2 - ||f-g||_2^2}{4} + \frac{||f-ig||_2^2 - ||f+ig||_2^2}{4i}.$$

We leave the proof of Lemma 2, which is computational, as well as those of the following two Lemmas to the reader. The way how the Fourier transform on $L^2(\mathbb{R})$ is defined, allows generalization of many properties of the Fourier transform on $L^1(\mathbb{R})$. We only list some in the following lemmas, the reader is encouraged to explore and find more.

Lemma 3. *For any $f, g \in L^2(\mathbb{R})$, $\int_{-\infty}^{\infty} f(x)\hat{g}(x)dx = \int_{-\infty}^{\infty} \hat{f}(x)g(x)dx$. In other words,*

$$\langle f, \overline{\hat{g}}\rangle = \langle \hat{f}, \overline{g}\rangle.$$

For any $f : \mathbb{R} \longrightarrow \mathbb{C}$, we can define the function f^- by $f^-(x) = f(-x)$. When $f \in L^1(\mathbb{R})$, it can be shown that $\widehat{(f^-)} = (\hat{f})^-$ and $\overline{\hat{f}} = \widehat{((\overline{f})^-)}$. They are also true in $L^2(\mathbb{R})$. Since $\overline{f^-} = (\overline{f})^-$ obviously always holds regardless, we do not include it in the lemma below, though it is also used in the last theorem of this chapter.

Lemma 4. *For any $f \in L^2(\mathbb{R})$, define f^- as $f^-(x) = f(-x)$. Then*

$$\widehat{(f^-)} = (\hat{f})^-, \overline{\hat{f}} = \widehat{((\overline{f})^-)}.$$

Theorem 4. *For any $g \in L^2(\mathbb{R})$, there exists a unique $f \in L^2(\mathbb{R})$, such that $\hat{f} = g$.*

Proof. By looking at the Inverse Fourier transform defined in the last chapter, we could guess that the correct way of expressing the "inverse Fourier transform" of g should be $\frac{1}{2\pi}\widehat{(g^-)}$. First we try to show that if we take $f(x) = \frac{1}{2\pi}\widehat{(g^-)}(x)$, then $\hat{f} = g$. To this end, we only need to show that $||g - \hat{f}||_2^2 = 0$. Indeed,

$$||g - \hat{f}||_2^2 = ||g||_2^2 - 2Re\langle g, \hat{f}\rangle + ||\hat{f}||_2^2 = ||g||_2^2 - 2Re\langle g, \overline{\widehat{((\overline{f})^-)}}\rangle + ||\hat{f}||_2^2$$
$$= ||g||_2^2 - 2Re\langle \hat{g}, \overline{((\overline{f})^-)}\rangle + ||\hat{f}||_2^2 = ||g||_2^2 - 2Re\langle \hat{g}, f^-\rangle + ||\hat{f}||_2^2$$
$$= ||g||_2^2 - 2Re\langle \hat{g}, \frac{1}{2\pi}\hat{g}\rangle + ||\hat{f}||_2^2 = \frac{1}{2\pi}||\hat{g}||_2^2 - \frac{2}{2\pi}||\hat{g}||_2^2 + 2\pi||f||_2^2$$
$$= -\frac{1}{2\pi}||\hat{g}||_2^2 + \frac{1}{2\pi}||\widehat{g^-}||_2^2 = 0.$$

The proof of uniqueness of such f is left to the reader. $\square$